Derivative Thomae formula for singular half-periods

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A complete generalization of Thomae theorems [1] in hyperelliptic case is obtained, that is values at zero of the lowest non-vanishing derivatives of theta functions with singular characteristics of arbitrary multiplicity are expressed in terms of branch points $\{e_i\}_{i=1}^{2g+2}$ and period matrix ω.

Theorem 1. *Let $\mathcal{I}_\mathfrak{m}\cup\mathcal{J}_\mathfrak{m}$ with $\mathcal{I}_\mathfrak{m}=\{i_1,\,\dots,\,i_{g+1-2\mathfrak{m}}\}$ and $\mathcal{J}_1=\{j_1,\,\dots,\,j_{g+1+2\mathfrak{m}}\}$ be a partition of the set of indices of all $2g+2$ branch points of hyperelliptic curve, and $[\mathcal{I}_\mathfrak{m}]$ denotes a singular characteristic of multiplicity $\mathfrak{m}$ corresponding to $\mathcal{A}(\mathcal{I}_\mathfrak{m})-K$. Let $\Delta(\mathcal{I}_\mathfrak{m})$ and $\Delta(\mathcal{J}_\mathfrak{m})$ be Vandermonde determinants built from $\{e_i\mid i\in\mathcal{I}_\mathfrak{m}\}$ and $\{e_j\mid j\in\mathcal{J}_\mathfrak{m}\}$. Then with a set $\mathcal{K}\subset\mathcal{J}_\mathfrak{m}$ of cardinality $\mathfrak{k}=2\mathfrak{m}-1$ or $2\mathfrak{m}$ the following relation holds*

$$\frac{\partial}{\partial v_{n_1}}\cdots\frac{\partial}{\partial v_{n_\mathfrak{m}}}\theta[\mathcal{I}_\mathfrak{m}](v)\big|_{v=0}=\epsilon\left(\frac{\det\omega}{\pi^g}\right)^{1/2}\Delta(\mathcal{I}_\mathfrak{m})^{1/4}\Delta(\mathcal{J}_\mathfrak{m})^{1/4}\times \\ \times\sum_{\substack{p_1,\dots,p_\mathfrak{m}\in\mathcal{K}\\ \text{all different}}}\prod_{i=1}^{\mathfrak{m}}\frac{\sum_{j=1}^{g}(-1)^{j-1}s_{j-1}(\mathcal{I}_\mathfrak{m}\cup\mathcal{K}^{(p_i)})\omega_{jn_i}}{\prod_{k\in\mathcal{K}\backslash\{p_1,\dots,p_\mathfrak{m}\}}(e_{p_i}-e_k)}. \tag{1}$$

where $s_j(\mathcal{I})$ denotes an elementary symmetric polynomial of degree j in branch points with indices from $\mathcal{I}$, and $\mathcal{K}^{(p_i)}=\mathcal{K}\backslash\{p_i\}$, and ϵ satisfies $\epsilon^8=1$.

Theta function with characteristic $[\varepsilon]$ is defined by the formula

$$\theta[\varepsilon](v;\tau)=\exp\big(\mathrm{i}\pi(\varepsilon'^t/2)\tau(\varepsilon'/2)+2\mathrm{i}\pi(v+\varepsilon/2)^t\varepsilon'/2\big)\theta(v+\varepsilon/2+\tau\varepsilon'/2;\tau). \tag{2}$$

All half-integer characteristics are represented by partitions of $2g+2$ indices of the form $\mathcal{I}_m\cup\mathcal{J}_m$ with $\mathcal{I}_m=\{i_1,\,\dots,\,i_{g+1-2m}\}$ and $\mathcal{J}_m=\{j_1,\,\dots,\,j_{g+1+2m}\}$, where m runs from 0 to $[(g+1)/2]$, and $[\cdot]$ means the integer part. Number m is called *multiplicity*. Infinity with index $2g+2$ is usually omitted in the sets. The characteristic $[\mathcal{I}_m]$ corresponds to partition $\mathcal{I}_m\cup\mathcal{J}_m$ in the following way

$$\sum_{i\in\mathcal{I}_m}\mathcal{A}(e_i)-K=\varepsilon(\mathcal{I}_m)/2+\tau\varepsilon'(\mathcal{I}_m)/2,$$

where K denotes the vector of Riemann constants. According to Riemann theorem $\theta(v+\mathcal{A}(\mathcal{I}_m)-K)$ vanishes to order m at $v=0$, Characteristics of multiplicity 0 and 1 are called non-singular even and odd, respectively. All other characteristics are called *singular*.

Some further results are derived from Theorem 1.

Corollary 2. *Let $\mathcal{I}_2\cup\mathcal{J}_2$ with $\mathcal{I}_2=\{i_1,\,\dots,\,i_{g-\mathfrak{k}}\}$ and $\mathcal{J}_2=\{j_1,\,\dots,\,j_{g+1+\mathfrak{k}}\}$, where $\mathfrak{k}=3$ or 4, be a partition of the set of $2g+1$ indices of finite branch points, such that singular characteristic $[\mathcal{I}_2]$, corresponding to $\mathcal{A}(\mathcal{I}_2)-K$, has multiplicity 2. Let $\Delta(\mathcal{I}_2)$ and $\Delta(\mathcal{J}_2)$ be Vandermonde determinants built from $\{e_i\mid i\in\mathcal{I}_2\}$ and $\{e_j\mid j\in\mathcal{J}_2\}$. Then*

$$\frac{\partial}{\partial v_{n_1}}\frac{\partial}{\partial v_{n_2}}\theta[\mathcal{I}_2](v)\big|_{v=0}=\epsilon\left(\frac{\det\omega}{\pi^g}\right)^{1/2}\Delta(\mathcal{I}_2)^{1/4}\Delta(\mathcal{J}_2)^{1/4}\sum_{i,j=1}^{g}(\hat{S}[\{\mathcal{I}_2\}])_{i,j}\omega_{in_1}\omega_{jn_2} \tag{3}$$

with $g\times g$ matrix

$$(\hat{S}[\mathcal{I}_2])_{i,j}=(-1)^{i+j}\Big(2s_{i-\mathfrak{k}+1}(\mathcal{I}_2)s_{j-\mathfrak{k}+1}(\mathcal{I}_2)-s_{i-\mathfrak{k}+2}(\mathcal{I}_2)s_{j-\mathfrak{k}}(\mathcal{I}_2)-s_{i-\mathfrak{k}}(\mathcal{I}_2)s_{j-\mathfrak{k}+2}(\mathcal{I}_2)\Big), \tag{4}$$

where ϵ satisfies $\epsilon^8 = 1$, and elementary symmetric functions $s_l(\mathcal{I}_2)$ are replaced by zero when $l < 0$.

Theorem 3. *For hyperelliptic curves of genera $g \geqslant 3$, when characteristics of multiplicity 2 exist, rank of every matrix of second derivative theta constants equals three, that is*

$$rank\big(\partial_v^2\theta[\mathcal{I}_2]\big) = 3.$$

Therefore, $\det\big(\partial_v^2\theta[\mathcal{I}_2]\big) = 0$ *in genera $g > 3$.*

Conjecture 4. *With a characteristic $[\mathcal{I}_\mathfrak{m}]$ of multiplicity $\mathfrak{m}$ corresponding to a partition $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m}$ with $\mathcal{I}_\mathfrak{m} = \{i_1, \ldots, i_{g-\mathfrak{k}}\}$ and $\mathcal{J}_\mathfrak{m} = \{j_1, \ldots, j_{g+1+\mathfrak{k}}\}$, where $\mathfrak{k} = 2\mathfrak{m} - 1$ or $2\mathfrak{m}$, of indices of $2g + 1$ finite branch points the following holds*

$$\partial_u^\mathfrak{m}\theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)\big|_{u=0} = \epsilon\left(\frac{\det\omega}{\pi^g}\right)^{1/2}\Delta(\mathcal{I}_\mathfrak{m})^{1/4}\Delta(\mathcal{J}_\mathfrak{m})^{1/4}\hat{S}[\mathcal{I}_\mathfrak{m}], \tag{5}$$

where u are non-normalized variables, and order $\mathfrak{m}$ tensor $\hat{S}[\mathcal{I}_\mathfrak{m}]$ belongs to the $\mathfrak{m}$-th tensor power $\mathcal{S}_{2\mathfrak{m}-1}^{\otimes\mathfrak{m}}$ of the vector space $\mathcal{S}_{2\mathfrak{m}-1}$ spanned by $2\mathfrak{m}-1$ vectors $\mathbf{s}_0$, $\mathbf{s}_1$, $\ldots$, $\mathbf{s}_{2\mathfrak{m}-2}$ such that $\mathbf{s}_d = \big(s_{j-\mathfrak{k}+d}(\mathcal{I}_\mathfrak{m})\big)_{j=1}^g$. The basis spanning $\hat{S}(\mathcal{I}_\mathfrak{m})$ could be found from partitions of $\mathfrak{m}(\mathfrak{m}-1)$ of length $\mathfrak{m}$ formed from numbers $\{0, 1, \ldots, 2\mathfrak{m} - 2\}$.

As a byproduct a generalization of Bolza formulas [2] are deduced.

Proposition 5. *Let $\mathcal{I}_\mathfrak{m}$ be a set of $g - \mathfrak{k}$ indices, and $\mathfrak{m} = [(\mathfrak{k} + 1)/2]$. Elementary symmetric polynomials in branch points $\{e_i \mid i \in \mathcal{I}_\mathfrak{m}\}$ of genus g hyperelliptic curve with period matrix ω are defined by*

$$s_j(\mathcal{I}_\mathfrak{m}) = (-1)^j\frac{\partial^\mathfrak{m}_{u_{2\mathfrak{k}-4(\mathfrak{m}-1)-1},\ldots,u_{2\mathfrak{k}-5},u_{2\mathfrak{k}+2j-1}}\theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)}{\partial^\mathfrak{m}_{u_{2\mathfrak{k}-4(\mathfrak{m}-1)-1},\ldots,u_{2\mathfrak{k}-5},u_{2\mathfrak{k}-1}}\theta[\mathcal{I}_\mathfrak{m}](\omega^{-1}u)}\Bigg|_{u=0}.$$

In particular,

$$e_\iota = -\frac{\partial^{[g/2]}_{u_{2(g mod 2)+1},\ldots,u_{2g-7},u_{2g-1}}\theta[\{\iota\}](\omega^{-1}u)}{\partial^{[g/2]}_{u_{2(g mod 2)+1},\ldots,u_{2g-7},u_{2g-3}}\theta[\{\iota\}](\omega^{-1}u)}\Bigg|_{u=0}.$$

Here $u = \omega v$ are non-normalized coordinates of Jacobian of the curve.

References

[1] Carl J. Thomae. Beitrag zur Bestimmung von $\theta(0, 0, \ldots 0)$ durch die Klassenmuduln algebraischer Functionen, *J. reine angew. Math.*, 71 : 201–222, 1870.

[2] Oscar Bolza, Uber die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades, *Math. Ann.*, 28(3) : 447–456, 1887.